

Multi-setting tripartite GHZ theorem

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Basing on our earlier work arXiv:1303.5326 we propose new version of the GHZ theorem which for higher dimensional systems (quDits). The new version of the theorem involves many local settings.

I. INTRODUCTION

Bell showed that a statistical inequality, a constraint of correlation that two remote systems should obey due to local realism, is violated by quantum mechanics for a signet state of two spin-1/2 particles [1]. Since Bell's work, many studies have been generalized to various cases [1–3]. Extending Bell's theorem is important not only for a deeper understanding of foundations of quantum mechanics. It allows developing new applications in quantum information processing, such as quantum cryptography, secret sharing, quantum teleportation, reduction of communication complexity, quantum key distribution and random numbers generation [4–8]. A Bell-type test is also regarded as a fundamental method to verify entanglement among subsystems. Therefore, search for extensions of Bell's theorem to more complex systems such as multipartite and/or high-dimensional systems is an important task in quantum information science [3].

Even though Bell's theorem was studied mostly in terms of statistical inequalities, different version of it *without inequalities*, was also shown for a multiqubit systems by Greenberger, Horne, and Zeilinger (GHZ) [9]. They derived an *all-versus-nothing* contradiction based on perfect correlations of the so-called GHZ states. This leads to a direct refutation of EPR ideas on relation between locality and elements of reality with quantum mechanics.

An all-versus-nothing test, that we call GHZ theorem, has been generalized to higher dimensional systems. For the sake of convenience, we shall use the tuple (N, M, D) to denote N parties, M measurements for each party and D distinct outcomes for each measurement. In Ref. [10], GHZ theorem was derived for a $(D+1, 2, D)$ problem. A probabilistic but conclusive GHZ-like test was shown for $(D, 2, D)$ in Ref. [11]. The (N, M, D) problem for odd $N > D$, $M = 2$, and even D was studied using operator relations by Cerf *et al.* [12]. Recently, Lee *et al.* employed incompatible composite observables for arbitrary (odd $N, 2$, even D) problems [13]. The conventional approaches are based on compatible observables.

All studies mentioned above have considered only two settings, i.e., each party choose two alternative observables. Here, we investigate a generalized GHZ theorem which involves more than 2 observables per party. To this end, we employ concurrent composite observables. Such observables are mutually incompatible but

still have a common eigenstate, here a generalized GHZ state. To construct more than 2 observables, we apply a phase shift operation with different phase values. The observables we used here can be realized by using multipoint beam splitters and phase shifters, as it is shown in Refs. [10, 13, 14]. We also propose a specific graph to show GHZ contradiction. First, we illustrate our principal idea with 3-setting GHZ theorem.

II. CONCURRENT OBSERVABLE

Some sets of observables have a common eigenstate. If a system is prepared in the eigenstate, the measurement results for such observables are concurrently appearing with certainty. Such observables are called “concurrent” [13]. For a quantum system of dimension D , greater than two, consider two Hermitian operators \hat{A} and \hat{B} , such that $\hat{A} = a|\psi\rangle\langle\psi| + \hat{A}'$ and $\hat{B} = b|\psi\rangle\langle\psi| + \hat{B}'$. The state $|\psi\rangle$ is then a common eigenstate of both observables as $\hat{A}(\hat{B})|\psi\rangle = a(b)|\psi\rangle$, even if $[\hat{A}, \hat{B}] = [\hat{A}', \hat{B}'] \neq 0$. Note that compatible observables are clearly concurrent.

To construct concurrent observables, in a Hilbert space \mathcal{H} , one can use the method of Ref. [13]. Consider a unitary operator \hat{U} , which is of the form of $\hat{U} = e^{i\phi}|\psi\rangle\langle\psi| + \hat{U}^\perp$ with $\hat{U}^\perp|\psi\rangle = 0$. Here \hat{U}^\perp is a unitary operator on a space \mathcal{H}^\perp which is defined by the requirement $\mathcal{H} = \mathcal{H}^\psi \oplus \mathcal{H}^\perp$, where \mathcal{H}^ψ is the one-dimensional space containing $|\psi\rangle$. Every such unitary operator leaves the state $|\psi\rangle$ unchanged, up to a global phase. If the state $|\psi\rangle$ satisfies $\hat{A}|\psi\rangle = \lambda|\psi\rangle$, then all transformed operators $\hat{B}_U = \hat{U}\hat{A}\hat{U}^\dagger$ are concurrent with \hat{A} .

We shall use the following local observables. Consider a measurement in D -dimensional Hilbert space represented by $\hat{A} = \sum_{n=0}^{D-1} \omega^n |n\rangle_{AA} \langle n|$, where $\omega = \exp(2\pi i/D)$. The operator \hat{A} is unitary, however one can uniquely relate it with a Hermitian observable \hat{H} , by requiring $\hat{A} = \exp(i\hat{H})$. Therefore the complex eigenvalues of \hat{A} , which could be called a unitary observable, can be associated with the measurement results, denoted by real eigenvalues of \hat{H} [12, 13, 15]. Such a unitary representation leads to a simplification of the mathematics without changing any physical results.

Based on above representation, we obtain the observable operator \hat{X} by applying quantum Fourier transformation \hat{F} on a reference unitary observable $\hat{Z} =$

$$\sum_{n=0}^{D-1} \omega^n |n\rangle\langle n|:$$

$$\hat{X} = \hat{F} \hat{Z} \hat{F}^\dagger = \sum_{n=0}^{D-1} |n\rangle\langle n+1|, \quad (1)$$

where $|n\rangle \equiv |n \bmod D\rangle$ and the eigenvector of \hat{X} is given by

$$|n\rangle_x = \hat{F}|n\rangle = \frac{1}{\sqrt{D}} \sum_{m=0}^{D-1} \omega^{-nm} |m\rangle. \quad (2)$$

The operator \hat{X} shifts a basis state periodically: $|n\rangle \rightarrow |n+1\rangle$ and $|D-1\rangle \rightarrow |0\rangle$.

In a similar manner, we construct another observables by using the phase shifter $\hat{P}_\alpha = \sum_n \omega^{\alpha n} |n\rangle\langle n|$. Then, we obtain the operator $\hat{X}(\alpha)$ by applying the \hat{P}_α on \hat{X} as $\hat{X}(\alpha) = \hat{P}_\alpha \hat{X} \hat{P}_\alpha^\dagger$:

$$\hat{X}(\alpha) = \omega^{-\alpha} \left(\sum_{n=0}^{D-2} |n\rangle\langle n+1| + \omega^{\alpha D} |D-1\rangle\langle 0| \right). \quad (3)$$

The observable $\hat{X}(\alpha)$ also performs a shift operation with additional phase on a given basis:

$$\hat{X}(\alpha)|n\rangle = \begin{cases} \omega^{-\alpha} |n-1\rangle & n \neq 0 \\ \omega^{\alpha(D-1)} |D-1\rangle & n = 0. \end{cases} \quad (4)$$

For each eigenvalue ω^n , the eigenvector of the observable $\hat{X}(\alpha)$ is given by

$$|n\rangle_\alpha = \hat{P}|n\rangle_x = \frac{1}{\sqrt{D}} \sum_{m=0}^{D-1} \omega^{(n+\alpha)m} |m\rangle. \quad (5)$$

To construct M -setting, we choose the phase value α in the set $\mathcal{P} = \{0, 1/M, \dots, (M-1)/M\}$. For $M = 2$, the operator $\hat{X}(\alpha)$ corresponds to the Pauli matrices $\hat{X}(0) = \hat{\sigma}_x$ and $\hat{X}(1/2) = \hat{\sigma}_y$.

Note that if α is an integer, the measurement basis set $\{|n\rangle_\alpha\}$ of $\hat{X}(\alpha)$ will be the same as $\{|n\rangle_x\}$ of \hat{X} except the ordering, i.e. $|n\rangle_\alpha = |n+\alpha\rangle_x$, and then $\hat{X}(\alpha) = \omega^{-\alpha} \hat{X}$. That is, the observable $\hat{X}(\alpha)$ is equivalent to \hat{X} , up to a phase factor $\omega^{-\alpha}$. Let $|n\rangle_\alpha$ be the eigenstate of $\hat{X}(\alpha)$ associated with eigenvalue ω^n , and $|m\rangle_\beta$ the eigenstate of $\hat{X}(\beta)$ associated with ω^m . If and only if α differs from β by an integer, then two measurement bases satisfy $|\alpha\rangle_\alpha \langle n|m\rangle_\beta|^2 = \delta_D(\gamma)$, where $\gamma = m - n + \beta - \alpha$. Here $\delta_D(\gamma) = 1$ if γ is congruent to zero modulo D and otherwise $\delta_D(\gamma) = 0$. That is, unless $\beta - \alpha$ is an integer, two local observables $\hat{X}(\alpha)$ and $\hat{X}(\beta)$ are inequivalent.

III. 3-SETTING GHZ THEOREM

We illustrate our idea by considering first a three quDits system. Take a 3 quDit GHZ state (D is an

integral multiple of 3)

$$|\psi\rangle = \frac{1}{\sqrt{D}} \sum_{n=0}^{D-1} |n, n, n\rangle. \quad (6)$$

The quDits are distributed to three sufficiently separated parties. Each party performs one of three non-degenerate local measurements on his or her quDit, each of which produces distinguishable D outcomes. The eigenvalues of the observables are of the form ω^k .

To show GHZ theorem, we employ three local observables $\hat{X}(0)$, $\hat{X}(1/3)$ and $\hat{X}(2/3)$ from Eq. (3). For the sake of convenience, we denote $\hat{X} = \hat{X}(0)$, $\hat{Y} = \hat{X}(1/3)$ and $\hat{Z} = \hat{X}(2/3)$. Then, the following concurrent observables are the eigenstates of the GHZ state as

$$\begin{aligned} \hat{X} \otimes \hat{X} \otimes \hat{X} |\psi\rangle &= |\psi\rangle, \\ \hat{X} \otimes \hat{Y} \otimes \hat{Z} |\psi\rangle &= \omega^{-1} |\psi\rangle, \\ \hat{X} \otimes \hat{Z} \otimes \hat{Y} |\psi\rangle &= \omega^{-1} |\psi\rangle, \\ \hat{Y} \otimes \hat{X} \otimes \hat{Z} |\psi\rangle &= \omega^{-1} |\psi\rangle, \\ \hat{Y} \otimes \hat{Z} \otimes \hat{X} |\psi\rangle &= \omega^{-1} |\psi\rangle, \\ \hat{Z} \otimes \hat{X} \otimes \hat{Y} |\psi\rangle &= \omega^{-1} |\psi\rangle, \\ \hat{Z} \otimes \hat{Y} \otimes \hat{X} |\psi\rangle &= \omega^{-1} |\psi\rangle, \\ \hat{Y} \otimes \hat{Y} \otimes \hat{Y} |\psi\rangle &= \omega^{-1} |\psi\rangle, \\ \hat{Z} \otimes \hat{Z} \otimes \hat{Z} |\psi\rangle &= \omega^{-2} |\psi\rangle. \end{aligned} \quad (7)$$

Local realistic theories assume that the outcomes of the measurements are predetermined, before the actual measurements. This means that the values of local realistic predictions for the correlations (7), for each experimental run, must satisfy:

$$\begin{aligned} \omega^{x_1} \omega^{x_2} \omega^{x_3} &= 1, \\ \omega^{x_1} \omega^{y_2} \omega^{z_3} &= \omega^{-1}, \\ \omega^{x_1} \omega^{z_2} \omega^{y_3} &= \omega^{-1}, \\ \omega^{y_1} \omega^{x_2} \omega^{z_3} &= \omega^{-1}, \\ \omega^{y_1} \omega^{z_2} \omega^{x_3} &= \omega^{-1}, \\ \omega^{z_1} \omega^{x_2} \omega^{y_3} &= \omega^{-1}, \\ \omega^{z_1} \omega^{y_2} \omega^{x_3} &= \omega^{-1}, \\ \omega^{y_1} \omega^{y_2} \omega^{y_3} &= \omega^{-1}, \\ \omega^{z_1} \omega^{z_2} \omega^{z_3} &= \omega^{-2}, \end{aligned} \quad (8)$$

where ω^{k_i} , $k \in \{x, y, z\}$ is an outcome of each measurement for i -th party.

In order to show GHZ contradiction, we propose a method which is based on a specific graph. All local realistic predictions such as Eqs. (8) are represented on the graph. We will show that a closed loop of the graph enables one to lead GHZ contradiction. We illustrate our idea by showing first a original GHZ contradiction. A 3-qubit GHZ state $|\Psi\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$ is an eigenstate of the following observables; $\hat{\sigma}_x \otimes \hat{\sigma}_y \otimes \hat{\sigma}_y$, $\hat{\sigma}_y \otimes \hat{\sigma}_x \otimes \hat{\sigma}_y$ and $\hat{\sigma}_y \otimes \hat{\sigma}_y \otimes \hat{\sigma}_x$ with the -1 eigenvalue and $\hat{\sigma}_x \otimes \hat{\sigma}_x \otimes \hat{\sigma}_x$

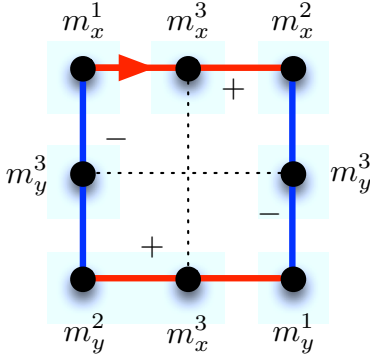


FIG. 1. A GHZ graph for 2-setting. Each vertex indicates a predetermined values and each solid line denotes the local realistic predictions such as $m_x^1 m_y^2 m_y^3 = m_y^1 m_x^2 m_y^3 = m_y^1 m_y^2 m_x^3 = -1$ and $m_x^1 m_x^2 m_x^3 = 1$. The points that are linked with a dot line have the same values.

with the unity eigenvalue, where $\hat{\sigma}_{x,y}$ are the Pauli matrices. On the other hand, local realistic predictions are given by $m_x^1 m_y^2 m_y^3 = m_y^1 m_x^2 m_y^3 = m_y^1 m_y^2 m_x^3 = -1$ and $m_x^1 m_x^2 m_x^3 = 1$. If one multiplies the first three equations as $m_x^1 m_x^2 m_x^3 (m_y^1 m_y^2 m_y^3)^2 = -1$, one can see this result is a contradiction with $m_x^1 m_x^2 m_x^3 = 1$.

Now, we propose a 2-setting GHZ graph which is depicted in Fig. 1. Each vertex indicates a predetermined value before the actual measurement and the vertices which are linked by dot lines have the same predetermined values. Each solid line implies the local realistic prediction as $m_x^1 m_y^2 m_y^3 = m_y^1 m_x^2 m_y^3 = m_y^1 m_y^2 m_x^3 = -1$ and $m_x^1 m_x^2 m_x^3 = 1$. Let us multiply each local realistic prediction along a clockwise loop where the thick red arrow indicates a starting point for loop. Note that in the multiplication we additionally multiply -1 by the local realistic prediction denoted by solid blue lines, as shown in Fig. 1. As a result, we obtain $(m_x^1 m_x^2 m_x^3)(m_y^1 m_x^2 m_y^3)(m_y^1 m_y^2 m_x^3)(m_x^1 m_y^2 m_y^3) = (m_x^1 m_x^2 m_x^3)^2 (m_y^1 m_y^2 m_y^3)^2 = -1$. We attain a contradiction.

Note that in order to prove GHZ contradiction we suggest a specific GHZ graph on which all local realistic predictions are represented. Multiplication them along a closed loop enables one to lead a GHZ contradiction.

Let us examine this method to 3-setting GHZ theorem, which has 9 local realistic predictions as shown in Eq. (8). In this case, the GHZ graph is shown in Fig. 2. As previously shown for the 2-setting GHZ graph, all local realistic predictions in Eq. (8) are represented by solid lines and each vertex indicates a predetermined value. A closed loop is denoted by colored thick lines and it begins from an arrow direction. The following equations are the

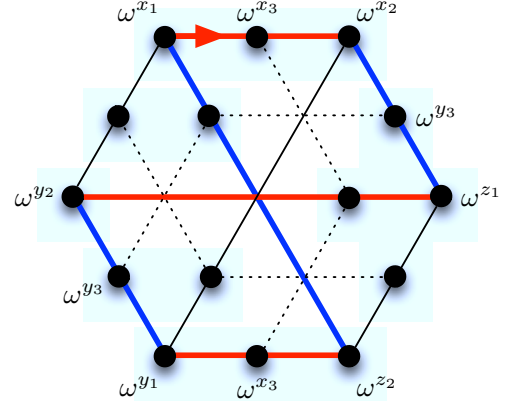


FIG. 2. A GHZ graph with 3-setting. Each vertex indicates a predetermined values before measurement. The points that are linked with a dot line imply that they have the same values. The thin black edges represent the local realistic prediction in Eq. (8). Additionally, when we multiply the local realistic predictions to see a contradiction, we do a complex conjugate to the predictions denoted by blue solid lines.

thick lines equations

$$\begin{aligned} \omega^{x1} \omega^{x2} \omega^{x3} &= 1, \\ \omega^{z1} \omega^{x2} \omega^{y3} &= \omega^{-1}, \\ \omega^{z1} \omega^{y2} \omega^{x3} &= \omega^{-1}, \\ \omega^{y1} \omega^{y2} \omega^{y3} &= \omega^{-1}, \\ \omega^{y1} \omega^{z2} \omega^{x3} &= \omega^{-1}, \\ \omega^{x1} \omega^{z2} \omega^{y3} &= \omega^{-1}. \end{aligned} \quad (9)$$

If one multiplies the local realistic predictions along the loop, one obtain $\omega^{3(x3-y3)-1} = 1$. In here, we do a complex conjugate to the local realistic predictions denoted by blue solid lines as $(\omega^{x1} \omega^{x2} \omega^{x3})(\omega^{-z1} \omega^{-x2} \omega^{-y3})(\omega^{z1} \omega^{y2} \omega^{x3})(\omega^{-y1} \omega^{-y2} \omega^{-y3})(\omega^{y1} \omega^{z2} \omega^{x3})(\omega^{-x1} \omega^{-z2} \omega^{-y3})$. As $\omega = \exp(2\pi i/D)$, if $D = 3d$ where d is an integer, an elementary algebra shows that there is no integer solution of $(x3 - y3) - 1 \equiv 0 \pmod{D}$. We obtain a 3-setting GHZ contradiction.

The GHZ contradiction we show here is a genuinely 3-setting one, as if one reduce the number of setting as 2, then for a given 3-quDit GHZ state, where $D = 3d$, one cannot see the GHZ contradiction.

Above method can be summarized as follows. First of all, let us go back the Eq. (8) and do following lists:

1. Choose two predetermined values of 3rd party, e.g., ω^{x3} and ω^{y3} . Then, we obtain 6 local realistic predictions with selected two predetermined values.
2. Multiply these 6 equations as follows: (3 equations with ω^{x3}) \times ComplexConjugate(3 equations with ω^{y3}).
3. Finally, we obtain $\omega^{3(x3-y3)} - 1 = 1$ and for $D = 3d$ there is no solutions. This is a GHZ contradiction.

With this method we can prove D -setting, where D is prime, tripartite GHZ contradiction.

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- [1] J. S. Bell, *Physics* **1**, 1 (1964).
 - [2] J. F. Clauser, M. A. Horne, A. Shimony, and R. A. Holt, *Phys. Rev. Lett.* **23**, 880 (1969).
 - [3] M. Żukowski and Č. Brukner, *Phys. Rev. Lett.* **88**, 210401 (2002); D. Collins, N. Gisin, N. Linden, S. Massar, and S. Popescu, *ibid.* **88**, 040404 (2002); W. Laskowski, T. Paterek, M. Żukowski, and Č. Brukner, *ibid.* **93**, 200401 (2004); W. Son, J. Lee, and M. S. Kim, *ibid.* **96**, 060406 (2006); J. Lim, J. Ryu, S. Yoo, C. Lee, J. Bang, and J. Lee, *New J. Phys.* **12**, 103012 (2010).
 - [4] A. Ekert, *Phys. Rev. Lett.* **67**, 661 (1991).
 - [5] R. Horodecki, M. Horodecki, and P. Horodecki, *Phys. Lett. A* **222**, 21 (1996).
 - [6] R. Cleve and H. Buhrman, *Phys. Rev. A* **56**, 1201 (1997); Č. Brukner, M. Żukowski, J.-W. Pan, and A. Zeilinger, *Phys. Rev. Lett.* **92**, 127901 (2004).
 - [7] M. Żukowski, A. Zeilinger, M. A. Horne, and H. Weinfurter, *Acta Phys. Pol. A* **93**, 187 (1998); M. Hillery, V. Bužek, and A. Berthiaume, *Phys. Rev. A* **59**, 1829 (1999); J. Kempe, *ibid.* **60**, 910 (1999); V. Scarani and N. Gisin, *Phys. Rev. Lett.* **87**, 117901 (2001); J. Barrett, L. Hardy, and A. Kent, *ibid.* **95**, 010503 (2005); A. Acín, N. Brunner, N. Gisin, S. Massar, S. Pironio, and V. Scarani, *ibid.* **98**, 230501 (2007).
 - [8] S. Pironio, A. Acín, S. Massar, A. B. de la Giroday, D. N. Matsukevich, P. Maunz, S. Olmschenk, D. Hayes, L. Luo, T. A. Manning, and C. Monroe, *Nature* **464**, 1021 (2010).
 - [9] D. M. Greenberger, M. A. Horne, and A. Zeilinger, in *Bell's Theorem, Quantum Theory, and Conceptions of the Universe*, edited by M. Kafatos (Kluwer, Dordrecht, 1989).
 - [10] M. Żukowski and D. Kaszlikowski, *Phys. Rev. A* **59**, 3200 (1999).
 - [11] D. Kaszlikowski and M. Żukowski, *Phys. Rev. A* **66**, 042107 (2002).
 - [12] N. J. Cerf, S. Massar, and S. Pironio, *Phys. Rev. Lett.* **89**, 080402 (2002).
 - [13] J. Lee, S.-W. Lee, and M. S. Kim, *Phys. Rev. A* **73**, 032316 (2006).
 - [14] J. Lee and S.-W. Lee, *J. Korean Phys. Soc.* **46**, 181 (2005).
 - [15] S.-W. Lee, Y. W. Cheong, and J. Lee, *Phys. Rev. A* **76**, 032108 (2007).